PHY5210 **Problem Set #3** **due 4 February 2015**

1. **(10 points):** A ray of light travels from point P_1 in a medium of refractive index n_1 to P_2 in a medium of index n_2, by way of the point Q on the plane interface between the two media, as in Figure 6.9 of Taylor. Show that Fermat's principle implies that, on the actual path followed, Q lies on the same vertical plane as P_1 and P_2 and obeys Snell's law, that $n_1 \sin\theta_1 = n_2 \sin\theta_2$. [*Hints:* Let the interface by the xz plane, and let P_1 lie on the y axis at $(0, h_1, 0)$ and P_2 in the xy plane at $(x_2, -h_2, 0)$. Finally let $Q = (x, 0, z)$. Calculate the time for the light to traverse the path P_1QP_2 and show that it is minimum when Q has $z = 0$ and satisfies Snell's law.]
2. **(10 points):** In many problems in the calculus of variations, you need to know the length ds of a short segment of a curve on a surface, as in Equation 6.1. Make a table giving the appropriate expression for ds in the following eight situations: (a) A curve given by $y = y(x)$ in a plane, (b) same but $x = x(y)$, (c) same but $r = r(\phi)$, (d) same but $\phi = \phi(r)$; (e) curve given by $\phi = \phi(z)$ on a cylinder of radius R, (f) same but $z = z(\phi)$; (g) curve given by $\theta = \theta(\phi)$ on a sphere of radius R, (h) same but $\phi = \phi(\theta)$.
3. **(5 points):** Consider a right circular cylinder or radius R centered on the z axis. Find the equation giving ϕ as a function of z for the geodesic (shortest path) on the cylinder between two points with cylindrical polar coordinates (R, ϕ_1, z_1) and (R, ϕ_2, z_2). Describe the geodesic. Is it unique? By imagining the surface of the cylinder unwrapped and laid out flat, explain why the geodesic has the form it does.
4. **(10 points):** Consider a medium in which the refractive index n is inversely proportional to r^2, that is, $n = a/r^2$ where r is the distance from the origin and a is a positive constant. Use Fermat's principle to find the path of a ray of light traveling in a plane containing the origin. [*Hint:* Use two dimensional polar coordinates and write the path as $\phi = \phi(r)$. The Fermat integral should have the form $\int f(\phi, \phi', r)dr$ where $f(\phi, \phi', r)$ is actually independent of ϕ. The Euler-Lagrange equation therefore reduces to $\partial f/\partial\phi' = \text{const}$. You can solve this for ϕ' and then integrate to give ϕ as a function of r. Rewrite this to give r as a function of ϕ and show that the resulting path is a circle through the origin. Discuss the progress of the light around the circle (how long does it take to reach the origin).]
5. **(5 points):** Consider a mass m moving in two dimensions with potential energy $U(x, y) = \frac{1}{2}kr^2$ where $r^2 = x^2 + y^2$. Write down the Lagrangian, using coordinates x and y, and find the two Lagrange equations of motion. Describe their solutions. [This is the potential energy of an ion in an "ion trap" which can be used to study the properties of individual atomic ions.]
6. **(10 points):** (a) Write down the Lagrangian $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2)$ for two particles of equal masses, $m_1 = m_2 = m$, confined to the x axis and connected by a spring with potential energy $U = \frac{1}{2}kx^2$. [Here x is the extension of the spring, $x = (x_1 - x_2 - \ell)$, where ℓ is the spring's unstretched length and I assume that mass 1 remains to the right of mass 2 at all times.] (b) Rewrite $\mathcal{L}$ in terms of the new variables $X = \frac{1}{2}(x_1 + x_2)$ (the CM position) and x (the extension), and write down the two equations of motion for X and x. (c) Solve for $X(t)$ and $x(t)$ and describe the motion.