PHY5210 **Problem Set #6** **due 2 March 2015**

1. **(10 points):** A thin rod (of width zero, but not necessarily uniform) is pivoted freely at one end about the horizontal y axis, being free to swing in the xz plane (x horizontal, z vertically down). Its mass is m, its CM is a distance a from the pivot, and its moment of inertia (about the y axis) is I. (a) Write down the equation of motion $\dot{L}_z = \Gamma_z$ and, assuming the motion is confined to small angles (measured from the downward vertical), find the period of this compound pendulum. ("Compound pendulum" is traditionally used to mean any pendulum whose mass is distributed — as contrasted with a "simple pendulum" whose mass is concentrated at a single point on a massless arm.)
(b) What is the length of the "equivalent" simple pendulum, that is, the simple pendulum with the same period?
2. **(10 points):** (a) If $\underline{\mathbf{I}}^{cm}$ denotes the moment of inertia tensor of a rigid body of mass M about its CM, and $\underline{\mathbf{I}}^{P}$ the corresponding tensor about a point P displaced from the CM by $\Delta\mathbf{r} = (\Delta x, \Delta y, \Delta z)$, prove that
$$I^P_{xx} = I^{cm}_{xx} + M(\Delta y^2 + \Delta z^2) \qquad \text{and} \qquad I^P_{yz} = I^{cm}_{yz} - M\Delta y \Delta z,$$
and so forth for the remaining diagonal and off-diagonal elements. This result generalizes the parallel-axis theorem that you probably learned in introductory physics, and means that once you know the inertia tensor for rotation about the CM, calculating it for any other origin is easy.
(b) Confirm the results of Example 10.2 (page 381) part (a) for the moment of inertia tensor of a cube rotating about a corner starting from the result of part (b) for the moment of inertia tensor of a cube rotating about its CM.
(c) Find the moment of inertia tensor of a uniform sphere rotating about a point on its surface. It is sometimes useful to view a rolling object (sphere, cylinder, or hoop) as instantaneously rotating about the point of contact with the surface.
3. **(5 points):** (a) Show that the principal moments of any rigid body satisfy $\lambda_3 \le \lambda_1 + \lambda_2$. *Hint:* Look at the integrals that define these moments. In particular, if $\lambda_1 = \lambda_2$, then $\lambda_3 \le 2\lambda_1$.
(b) For what shape of body is $\lambda_3 = \lambda_1 + \lambda_2$?
4. **(10 points):** (a) A rigid body is rotating freely, subject to zero torque. Use Euler's equations to prove that the magnitude of the angular momentum $\mathbf{L}$ is constant. (Multiply the ith equation by $L_i = \lambda_i \omega_i$ and add the three equations.)
(b) In much the same way, show that the kinetic energy of rotation $T_{\text{rot}} = \frac{1}{2}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2\right)$ is constant.
5. **(15 points):** An axially symmetric space station (principal axis $\mathbf{e}_3$, and $\lambda_1 = \lambda_2$) is floating in free space. It has rockets mounted symmetrically on either side that are firing and exert a constant torque Γ about the symmetry axis. Solve Euler's equations exactly for ω (relative to the body axis) and describe the motion. At $t = 0$ take $\omega = (\omega_{10}, 0, \omega_{30})$.
6. **(10 extra credit points):** A pencil is set spinning in an upright position. Show that it must spin at a rate
$$S > \sqrt{\frac{128ga}{b^4}\left(\frac{a^2}{3} + \frac{b^2}{16}\right)}$$
in order to remain upright under the influence of gravity. Assume that the pencil is a uniform cylinder of length a and diameter b that spins on a point on the symmetry axis (neglect the effect of the point on the CM or inertia tensor). Find the value of the spin in revolutions per second for $a = 10\,\text{cm}$ and $b = 1\,\text{cm}$.